Report on 1D rebound of a rod

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A simplified model for a deformable rod normally impacting a rigid plane an rebounding off of it.

Key words: Elasticity, impacts.

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1. Objectives

- determine the peak strain on the rod when it bounces off the floor.
- determine how the peak strain depends on the parameters describing the rod and the strength of the impact.

2. Problem data

A copper rod with length L, cross-sectional area A (not given but necessary for some considerations), density ρ, Young's modulus E and damping coefficient α is dropped form a height $h = 1\,\mathrm{m}$ (with $g = 9.8\,\mathrm{m/s^2}$).

3. Problem formulation

We define the location of each point on the rod as $\eta(z,t)$, where $z \in [0, L]$ is the initial undeformed location of the rod. Hence, the displacement of each point is given by

$$u(z,t) = \eta(z,t) - z. \tag{3.1}$$

In this conditions, the strain on the rod is

$$e = \partial_z u = \partial_z \eta(z,t) - 1, \tag{3.2}$$

and the normal stress (traction) in the rod is

$$\tau = Ee = E\left(\partial_z \eta(z,t) - 1\right). \tag{3.3}$$

Newton's second law of motion for the rod results in

$$\rho \partial_{tt} \eta = \partial_z \tau - \alpha \partial_{tz} e. \tag{3.4}$$

Where $\rho \partial_{tt} u$ corresponds to ma and broken down into the pressure applied to the the cross-section area (PA), against the internal resistance of the rod: $\alpha \partial_{tz} e$ is the internal friction of the rod which resists the force applied to the rod during the impact.

Substituting equations (3.3) into (3.4) we have

$$\partial_{tt} \eta = \frac{E}{\rho} \partial_z \left(\partial_z \eta(z,t) - 1\right) - \frac{\alpha}{\rho} \partial_{tz} e, \tag{3.5}$$

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and substituting equation 3.2, we have

$$\partial_{tt}\eta = \frac{E}{\rho}\partial_{zz}\eta(z,t) - \frac{\alpha}{\rho}\partial_{tz}\left(\partial_z\eta\left(z,t\right) - 1\right), \tag{3.6}$$

i.e.

$$\partial_{tt}\eta = \frac{E}{\rho}\partial_{zz}\eta(z,t) - \frac{\alpha}{\rho}\partial_{tzz}\eta\left(z,t\right), \tag{3.7}$$

This is a damped wave equation, for which we know the wave speed is

$$C = \sqrt{\frac{E}{\rho}}. \tag{3.8}$$

4. First simple insights

When a deformable object impacts a rigid surface, the force it receive from the surface must be so that the impulse it causes on the object is equal to the change in momentum of the portion of the object that has slowed to 0 velocity. It is only a portion of the object that slows down at each instant, because there is a finite velocity for the propagation of disturbances (the wave propagation speed).

We can therefore estimate that

$$\int_0^t f(\tau)d\tau \approx \rho ACtV_0, \tag{4.1}$$

where f is the vertical upward force exerted by the floor on the rod,

$$V_0 = \sqrt{2gh}, \tag{4.2}$$

and the right hand side of (4.1) is given by the mass of the portion of the object that has slowed to zero velocity times the change in velocity it underwent.

Taking the derivative with respect to time of equation (4.1), we have

$$f(t) \approx \rho ACV_0. \tag{4.3}$$

This calculation assumes that the impact speed of the rod was not greater than the wave propagation speed through it, which is a reasonable assumption for a typical solid rod falling from a height of one metre.

Moreover, it is clear that this is considering only the impulse needed to stop the rod, and that the force may be applied for longer (or with greater intensity) inducing further impulse to the rod (so that after the collision it moves upward); however, the contribution to make the rod stop is likely where the maximum forces occur and therefore where the maximum pressures and strains occur.

Furthermore, other factors might have a small contribution (such as the damping), but we are safe to assume these inertial balances will dominate the impact.

Now, since we are interested in the maximum strain, which in all likelihood happens at the leading end of the impacting rod, we can use equations (4.3) and (3.3) to obtain

$$Ee = P = \frac{f}{A} \approx \rho CV_0, \tag{4.4}$$

and therefore we would expect that the maximum strain be

$$e_{\text{max}} \approx \frac{\rho CV_0}{E}. \tag{4.5}$$

Using (4.2) and (3.8) to express this estimate as a function of the original data, we have

$$e_{\max} \approx \sqrt{\frac{2g\rho h}{E}}. \tag{4.6}$$

5. A more in-depth approach

At this point it is perhaps more convenient to formulate the problem in terms of the location of the points rather than in terms of their displacement.

For an arbitrary section of the rod, of original length δ_z we have

$$\begin{aligned}\rho A\delta_z\partial_{tt}\eta(z,t) = &-\rho g A\delta_z \\ &+ P(z,t)A \\ &- P(z+\delta_z,t)A \\ &- A\alpha\partial_{tz}\left[\eta(z,t)-z\right] \\ &+ A\alpha\partial_{tz}\left[\eta(z+\delta_z,t)-z\right]\end{aligned} \tag{5.1}$$

i.e.

$$\begin{aligned}\rho\partial_{tt}\eta(z,t) = &-\rho g \\ &- \frac{P(z+\delta_z,t)-P(z,t)}{\delta_z} \\ &+ \alpha\frac{\partial_{tz}\eta(z+\delta_z,t)-\partial_{tz}\eta(z,t)}{\delta_z},\end{aligned} \tag{5.2}$$

which implies

$$\rho\partial_{tt}\eta(z,t) = -\rho g - \partial_z P + \alpha\partial_{tzz}\eta, \tag{5.3}$$

which is subject to

$$\partial_z(\eta(z=0,t)-z) = -\frac{f(t)}{EA}, \tag{5.4}$$

$$\partial_z(\eta(z=1,t)-z) = 0, \tag{5.5}$$

$$\partial_t(\eta(z,t=0)-z) = -V_0, \tag{5.6}$$

$$\eta(z,t=0) = z. \tag{5.7}$$

6. Non-dimensionalisation

We take L, C and the mass of the rod ρAL as unit length, speed and mass, respectively, and we define the dimensionless numbers $G = gL/C^2$, $D = \alpha/(\rho CL)$ and $U = V_0/C$.

Moreover, we define $w(z,t) = \partial_t\eta(z,t)$, and we have We thus have the dimensionless problem

$$\partial_t\eta = w, \tag{6.1}$$

$$\partial_t w = -G + \partial_{zz}\eta + D\partial_{zz}w; \tag{6.2}$$

subject to

$$\partial_z\eta(z=0,t) = 1 - f(t), \tag{6.3}$$

$$\partial_z\eta(z=1,t) = 1, \tag{6.4}$$

$$\partial_z w(z=0,t) = -f'(t), \tag{6.5}$$

$$\partial_z w(z=1,t) = 0, \tag{6.6}$$

$$\eta(z,t=0) = z, \tag{6.7}$$

$$w(z,t=0) = -U \qquad \text{for } 0 \le z \le 1. \tag{6.8}$$

7. Numerical implementation

We use the backward Euler method in time, and second order finite differences in space to obtain

$$\eta_i^{k+1} - \delta_t w_i^{k+1} = \eta_i^k; \tag{7.1}$$

$$-\frac{\delta_t}{(\delta_z)^2}(\eta_{i-1}^{k+1} - 2\eta_i^{k+1} + \eta_{i+1}^{k+1}) + w_i^{k+1} - \frac{\delta_t D}{(\delta_z)^2}(w_{i-1}^{k+1} - 2w_i^{k+1} + w_{i+1}^{k+1}) = w_i^k - \delta_t G; \tag{7.2}$$

with the initial condition

$$\eta_i^1 = z_i, \tag{7.3}$$

$$w_i^1 = -U. \tag{7.4}$$

Defining "ghost points" for $i = 0$ and $i = n_z + 2$, we have

$$\frac{\eta_2^{k+1} - \eta_0^{k+1}}{2\delta_z} = 1 - f^{k+1}, \tag{7.5}$$

$$\frac{\eta_{n_z+2}^{k+1} - \eta_{n_z}^{k+1}}{2\delta_z} = 1, \tag{7.6}$$

$$\frac{w_2^{k+1} - w_0^{k+1}}{2\delta_z} = -\frac{f^{k+1} - f^k}{\delta_t}, \tag{7.7}$$

$$\frac{w_{n_z+2}^{k+1} - w_{n_z}^{k+1}}{2\delta_z} = 0. \tag{7.8}$$

From the equations above we have

$$\eta_2^{k+1} - \eta_0^{k+1} = 2\delta_z - 2\delta_z S f^{k+1}, \tag{7.9}$$

$$\eta_{n_z+2}^{k+1} - \eta_{n_z}^{k+1} = 2\delta_z, \tag{7.10}$$

$$w_2^{k+1} - w_0^{k+1} = -2\delta_z \frac{f^{k+1}}{\delta_t} + 2\delta_z \frac{f^k}{\delta_t}, \tag{7.11}$$

$$w_{n_z+2}^{k+1} - w_{n_z}^{k+1} = 0; \tag{7.12}$$

which yields

$$\eta_0^{k+1} = \eta_2^{k+1} - 2\delta_z + 2\delta_z f^{k+1}, \tag{7.13}$$

$$\eta_{n_z+2}^{k+1} = \eta_{n_z}^{k+1} + 2\delta_z, \tag{7.14}$$

$$w_0^{k+1} = w_2^{k+1} + 2\delta_z \frac{f^{k+1}}{\delta_t} - 2\delta_z \frac{f^k}{\delta_t}, \tag{7.15}$$

$$w_{n_z+2}^{k+1} = w_{n_z}^{k+1}. \tag{7.16}$$

This means that for $i = 1$ equation (7.2) is expressed as

$$-\frac{\delta_t}{(\delta_z)^2}(\eta_0^{k+1} - 2\eta_1^{k+1} + \eta_{i+1}^2) + w_1^{k+1} - \frac{\delta_t D}{(\delta_z)^2}(w_0^{k+1} - 2w_1^{k+1} + w_2^{k+1}) = w_1^k - \delta_t G; \tag{7.17}$$

and using (7.13) and (7.15) we have

$$\begin{aligned} &-\frac{\delta_t}{(\delta_z)^2}(\eta_2^{k+1} - 2\delta_z + 2\delta_z f^{k+1} - 2\eta_1^{k+1} + \eta_{k+1}^2) + w_1^{k+1} \\ &- \frac{\delta_t D}{(\delta_z)^2}(w_2^{k+1} + 2\delta_z \frac{f^{k+1}}{\delta_t} - 2\delta_z \frac{f^k}{\delta_t} - 2w_1^{k+1} + w_2^{k+1}) = w_1^k - \delta_t G; \end{aligned} \tag{7.18}$$

i.e.

$$\begin{aligned} &-\frac{\delta_t}{(\delta_z)^2}(\eta_2^{k+1} - 2\eta_1^{k+1} + \eta_2^{k+1}) + w_1^{k+1} - \frac{\delta_t D}{(\delta_z)^2}(w_2^{k+1} - 2w_1^{k+1} + w_2^{k+1}) \\ &- \frac{\delta_t}{(\delta_z)^2}(2\delta_z f^{k+1}) - \frac{\delta_t D}{(\delta_z)^2}(2\delta_z \frac{f^{k+1}}{\delta_t}) = w_1^k \\ &- \delta_t G + \frac{\delta_t}{(\delta_z)^2}(-2\delta_z) + \frac{\delta_t D}{(\delta_z)^2}(-2\delta_z \frac{f^k}{\delta_t}); \end{aligned} \tag{7.19}$$

or, equivalently,

$$\begin{aligned} &-\frac{\delta_t}{(\delta_z)^2}(2\eta_2^{k+1} - 2\eta_1^{k+1}) + w_1^{k+1} - \frac{\delta_t D}{(\delta_z)^2}(2w_2^{k+1} - 2w_1^{k+1}) - 2\frac{\delta_t}{\delta_z} f^{k+1} - \frac{2D}{\delta_z} f^{k+1} \\ &= w_1^k - \delta_t G - 2\frac{\delta_t}{\delta_z} - 2D\frac{\delta_z}{\delta_t} f^k. \end{aligned} \tag{7.20}$$

Similarly for $i = n_z + 1$ we have

$$-\frac{\delta_t}{(\delta_z)^2}(\eta_{n_z}^{k+1} - 2\eta_{n_z+1}^{k+1} + \eta_{n_z+2}^{k+1}) + w_i^{k+1} - \frac{\delta_t D}{(\delta_z)^2}(w_{n_z}^{k+1} - 2w_{n_z+1}^{k+1} + w_{n_z+2}^{k+1}) = w_i^k - \delta_t G; \tag{7.21}$$

and, using equation (7.14) and (7.16), we have

$$-\frac{\delta_t}{(\delta_z)^2}(\eta_{n_z}^{k+1} - 2\eta_{n_z+1}^{k+1} + \eta_{n_z}^{k+1} + 2\delta_z) + w_i^{k+1} - \frac{\delta_t D}{(\delta_z)^2}(w_{n_z}^{k+1} - 2w_{n_z+1}^{k+1} + w_{n_z}^{k+1}) = w_i^k - \delta_t G, \tag{7.22}$$

i.e.

$$-\frac{\delta_t}{(\delta_z)^2}(2\eta_{n_z}^{k+1} - 2\eta_{n_z+1}^{k+1}) + w_i^{k+1} - \frac{\delta_t D}{(\delta_z)^2}(2w_{n_z}^{k+1} - 2w_{n_z+1}^{k+1}) = w_i^k - \delta_t G + 2\frac{\delta_t}{\delta_z}, \tag{7.23}$$

7.1. *System of equations*

The finite difference equations above can be summarised in the following matrix equation

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} & 0 \\ \mathcal{C} & \mathcal{D} & \mathcal{E} \end{bmatrix} \begin{bmatrix} \eta^{k+1} \\ w^{k+1} \\ f^{k+1} \end{bmatrix} = \begin{bmatrix} \eta^k \\ w^k \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{F} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{G} \end{bmatrix}, \tag{7.24}$$

where $\mathcal{A}$ is the identity of size n_z+1, $\mathcal{B}=-\delta_t\mathcal{A}$,

$$\mathcal{C}=-\frac{\delta_t}{\delta_z^2}\begin{bmatrix} 2 & -2 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ 0 & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -2 & 2 \end{bmatrix}, \tag{7.25}$$

moreover $\mathcal{D}=A+D\mathcal{C}$,

$$\mathcal{E}=\begin{bmatrix} -\frac{2}{\delta_z}\left(\delta_t+D\right) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{7.26}$$

$$\mathcal{F}=-G\delta_t\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{7.27}$$

and

$$\mathcal{G}=\begin{bmatrix} -2\frac{\delta_t}{\delta_z}-2D\frac{\delta_z}{\delta_t}f^k \\ 0 \\ \vdots \\ 0 \\ 2\frac{\delta_t}{\delta_z} \end{bmatrix}. \tag{7.28}$$

This is a rectangular system that can be manipulated to solve the problem when there is contact as well as when the rod is in the air.

8. Results

Appendix A. Matlab code

```
%1D impact and rebound of a rod

close all
clear
clc

%Physical parameters
%(silicon rubber made 100 times less rigid)
rho = 1100; %Density of the rod in Kg/m^3
E = 1.5E7; %Young's modulus of the rod in N/m^2
L = .1; %Length of the rod in metres
alpha = .1; %Damping factor of the rod in Kg/(ms)
%(equivalent to dynamic viscosity)
A = .0001; %Cross-sectional area in m^2 (not given, but needed)
V_0 = sqrt(2*9.8*1); %Initial speed in metres per second
g = 9.8; %gravity in m/s^2
```

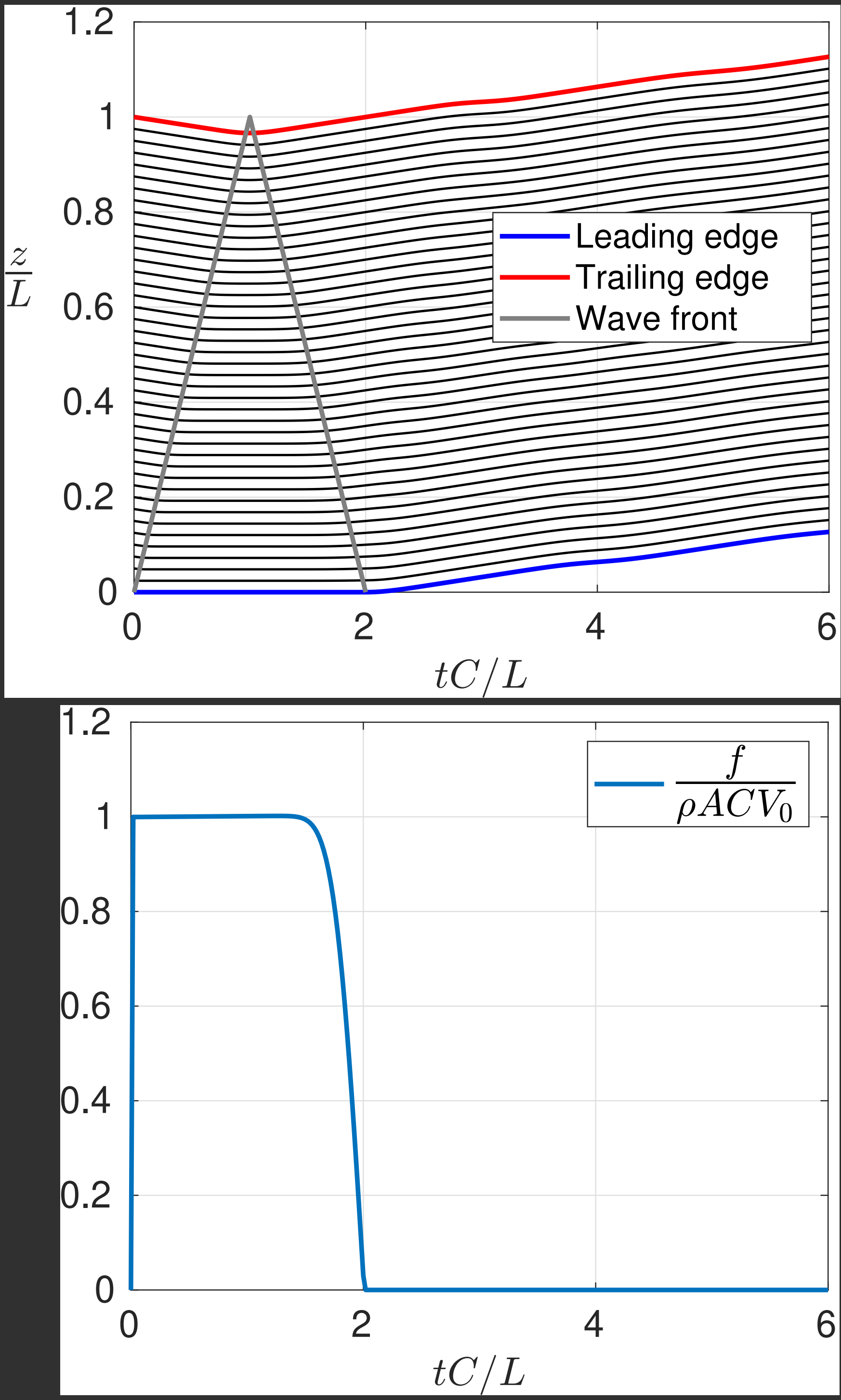

FIGURE 1. Caption

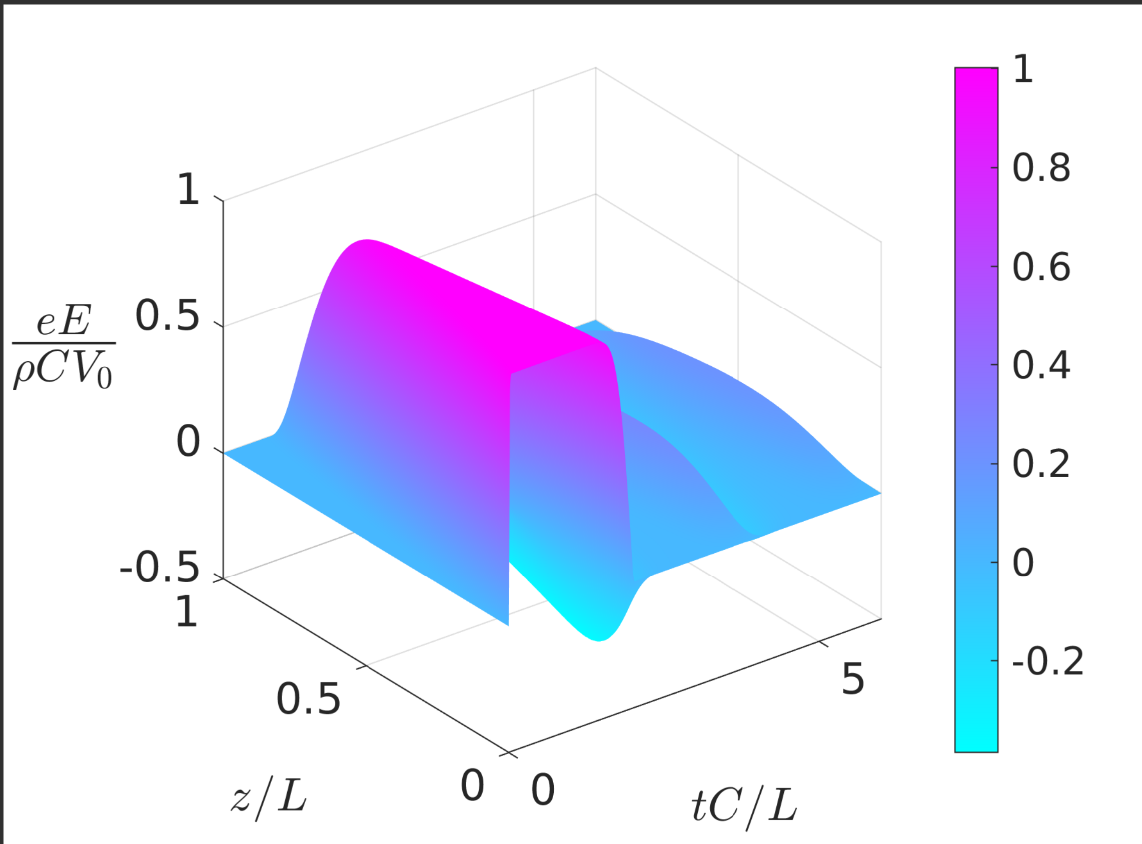

FIGURE 2. This simulation results in a maximum strain of $1.0023\rho C v_0/E$

```
%Numerical parameters
n_z = 4000; %number of segments defined on the rod
%ideally use multiple of 100 (for plotting)
n_t = 50; %number of time intervals that fit in one Unit_time
n_steps = 6*n_t; %initial choice of number of time-steps

%Base Units
Unit_length = L;
Unit_speed = sqrt(E/rho);
Unit_mass = rho*A*L;

%Derived units
Unit_time = Unit_length/Unit_speed;
Unit_force = Unit_mass*Unit_speed^2/Unit_length;
Unit_density = Unit_mass/(Unit_length^3);

%Time step
delta_t = 1/n_t;
delta_z = 1/n_z;
```

```
%Dimensionless numbers
DD = alpha/(rho*Unit_speed^2*Unit_time); %Dimensionless damping
GG = g*Unit_length/(Unit_speed^2);
UU = V_0/Unit_speed;

%Variables
eta = zeros(n_z+1,n_steps+1); %Location of points
w = zeros(n_z+1,n_steps+1); %Velocity of displacement
f = zeros(1,n_steps+1); %Force at the contact point

%Initial values
eta(:,1) = (0:delta_z:1);
w(:,1) = -UU*ones(n_z+1,1);

%Matrices
Mat_A = eye(n_z+1);
Mat_B = -delta_t*eye(n_z+1);
Mat_C = - (delta_t/(delta_z^2)) ...
          *(-2*diag(ones(n_z+1,1)) ...
            +  diag(ones(n_z,1),1) ...
            +  diag(ones(n_z,1),-1) ...
           );
Mat_C(1,2) = 2*Mat_C(1,2);
Mat_C(end,end-1) = 2*Mat_C(end,end-1);
Mat_D = eye(n_z+1) ...
        - DD ...
          *(delta_t/(delta_z^2)) ...
          *(-2*diag(ones(n_z+1,1)) ...
            +  diag(ones(n_z,1),1) ...
            +  diag(ones(n_z,1),-1) ...
           );
Mat_D(1,2) = 2*Mat_D(1,2);
Mat_D(end,end-1) = 2*Mat_D(end,end-1);
Mat_E = zeros(n_z+1,1);
Mat_E(1) = -2*DD/delta_z-2*delta_t/delta_z;

Mat_System = [Mat_A,Mat_B,zeros(n_z+1,1);Mat_C,Mat_D,Mat_E];

%main loop
for ind_time = 1:n_steps
    disp(ind_time/n_steps)
    b = [eta(:,ind_time);(w(:,ind_time)-GG*delta_t*ones(n_z+1,1))];
    b(n_z+2) = b(n_z+2)-2*DD*f(ind_time)*delta_z/delta_t-2*delta_t/delta_z;
    b(2*n_z+2) = b(2*n_z+2)+2*delta_t/delta_z;
    Mat_System = sparse(Mat_System);

    %First we solve without force
    sol_free = Mat_System(1:end,1:end-1)\b;
    if sol_free(1)<0
        sol_forced = Mat_System(2:end,[2:n_z+1,n_z+3:end])\b(2:end);
```

```
        eta(1,ind_time+1) = 0;
        w(1,ind_time+1) = 0;
        eta(2:end,ind_time+1) = sol_forced(1:n_z);
        w(2:end,ind_time+1) = sol_forced(n_z+1:2*n_z);
        f(ind_time+1) = sol_forced(end);
    else
        eta(:,ind_time+1) = sol_free(1:n_z+1);
        w(:,ind_time+1) = sol_free(n_z+2:2*n_z+2);
    end
end

%Plotting rod points
fig_a = figure
bottom = plot(delta_t*(0:n_steps),eta(1,:),'b','LineWidth',2);
hold on
top = plot(delta_t*(0:n_steps),eta(end,:),'r','LineWidth',2);
grid on
for ind_z = 101:100:n_z
    plot(delta_t*(0:n_steps),eta(ind_z,:),'k','LineWidth',1)
end
wave_front = plot([0 1 1 2],[0 1 1 0],'color',[.5 .5 .5],'LineWidth',2);
xlabel('$tC/L$','interpreter','latex')
set(gca,'FontSize',16)
ylabel('$\frac{z}{L}\ \ \ \ $','interpreter','latex','FontSize',24,'rotation',0)
legend([bottom, top, wave_front],'Leading edge','Trailing edge','Wave front')
print(fig_a,'-depsc','-r300','rod_motion.eps')

%Plotting forces
fig_b = figure;
plot(delta_t*(0:n_steps),f*Unit_force/(V_0*rho*A*Unit_speed),'LineWidth',2)
grid on
xlabel('$tC/L$','interpreter','latex')
ylabel('$\ \ \ \ \ \ \ \ $','interpreter','latex','FontSize',24,'rotation',0)
set(gca,'FontSize',16)
legend('$\frac{f}{\rho A C V_0} $','interpreter','latex','FontSize',24)
print(fig_b,'-depsc','-r300','force.eps')

%strain
strain = zeros(size(eta));
for ind_time = 1:n_steps+1
    strain(1,ind_time) = f(ind_time)*Unit_force/(E*A);
    for ind_z = 2:n_z
        strain(ind_z,ind_time) = 1 ...
                                -(  eta(ind_z+1,ind_time) ...
                                  - eta(ind_z-1,ind_time) ...
                                 ) ...
                                 /(2*delta_z);
    end
end
```

```
%Plotting strain
fig_c = figure;
surf(delta_t*(0:n_steps),delta_z*(0:n_z),strain*E/(rho*Unit_speed*V_0),'LineStyle',
colormap(cool)
xlabel('$tC/L$','interpreter','latex')
ylabel('$z/L$','interpreter','latex')
set(gca,'FontSize',16)
zlabel('$\frac{eE}{\rho C V_0}\ \ \ \ $','interpreter','latex','FontSize',24,'rotat
grid on
colorbar('EastOutside')
print(fig_c,'-depsc','-r300','strain.eps')

max(max(strain*E/(rho*Unit_speed*V_0)))
```